Quantum Sensors: Problems.

Limit cycles.

1. The van der Pol (VdP) oscillator model was introduced to explain limit cycles (sometimes called relxation oscillations) in electronic circuits. The equations of motion are

$$\begin{aligned}\dot{x} &= y \ , \\ \dot{y} &= \epsilon(1-x^2)y - x + a \ .\end{aligned}$$

   The original model set $a = 0$. If $a \neq 0$ it is called the driven VdP oscillator. when $\epsilon = 0$ this describes an undamped harmonic oscillator. When $\epsilon \neq 0$ it describes non linear damping for $x^2 > 1$ and non linear gain when $x^2 < 1$.

   Use Mathematica, Matlab or Python to simulate the dynamics of this system and investigate the fixed points and limit cycles as a function of ϵ, a. There is a nice Mathematica Notebook for here :`https://demonstrations.wolfram.com/VanDerPolOscillator/`.

2. A semiclassical model for a laser is given in terms of the complex amplitude of the field in the rotating frame. $\alpha(t)$

$$\dot{\alpha} = -\frac{\kappa\alpha}{2}\left(1 - \frac{Gn_s}{\kappa(|\alpha|^2 + n_s)}\right)$$

   Show that are two fixed points $\alpha_0 = 0$ and $|\alpha_0|^2 = G(n_s - 1)/\kappa$ and investigate their stability. This is done by using a linear approximation for the dynamics near each fixed point and calculating the eigenvalues of the corresponding matrix. The second solution is the above threshold limit-cycle solution.

   To find the fixed points set

$$\frac{\kappa\alpha}{2}\left(1 - \frac{Gn_s}{\kappa(|\alpha|^2 + n_s)}\right) = 0$$

3. The Photon number distribution, $p_n(t) = \langle n|\rho(t)|n\rangle$ for a laser model satisfies the Markov master equation,

$$\frac{dp_n(t)}{dt} = Gn_s\left[\frac{n}{n+n_s}p_{n-1} - \frac{n+1}{n+1+n_s}p_n\right] + \kappa(n+1)p_{n+1} - \kappa n p_n$$

(a) Show that the steady state solution is

$$p_n^{ss} = \mathcal{N}\frac{(Gn_s/\kappa)^{n+n_s}}{(n+n_s)!}$$

(b) Below threshold $G < \kappa$ show that this can be approximated by

$$p_n^{ss} \approx \frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n \qquad G < \kappa$$

(c) Show that this is analogous to a thermal distribution with mean photon number given by

$$\bar{n} = \frac{G}{\kappa - G}$$

(d) Show that well above threshold, $G >> \kappa$, the steady state distribution is Poisson with mean $\bar{n} = Gn_s/\kappa$

4. A quantum simple harmonic oscillator undergoing phase diffusion satisfies the master equation

$$\dot{\rho} = -i\omega[a^\dagger a, \rho] + \Gamma\mathcal{D}[a^\dagger a]\rho$$

where Γ is the quantum phase diffusion rate

(a) Show that the phase diffusion has no effect on the number state distribution.

(b) Derive an equation of motion for the mean amplitude $\langle a\rangle = \mathrm{tr}(a\rho(t))$ and show that the mean amplitudes decays at a rate determined by Γ.

5. the Hamiltonian for a two-level system driven harmonically is

$$H = \frac{\hbar\omega_a}{2}\sigma_z + \hbar\Omega\cos(\omega_L t + \phi)\sigma_x$$

$\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|$, $\sigma_y = -i(|e\rangle\langle g| - |g\rangle\langle e|)$

(a) Show that in the interaction picture the Hamiltonian is

$$H = \frac{\hbar\delta}{2}\sigma_z + \hbar\Omega(\cos\phi\ \sigma_x - \sin\phi\ \sigma_y)$$

where $\delta = \omega_a - \omega_L$.

(b) If the Initial state: $|\psi(0)\rangle = |g\rangle$ show that the probability to find system in excited state is

$$P_e(t) = \frac{\Omega^2}{\tilde{\Omega}^2}\sin^2\left[\frac{\tilde{\Omega}\ t}{2}\right]$$

where $\tilde{\Omega} = \sqrt{\Omega^2 + \delta^2}$,

Sensors

1. Consider Ramsey fringe interferometry for which the N atoms are prepared in the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{N}}|ggg\ldots g\rangle + |eee\ldots e\rangle)$$

where $|ggg\ldots g\rangle = |g\rangle \otimes |g\rangle \otimes |g\rangle \otimes \ldots |g\rangle$ and likewise for the excited states. Show that there is a $N^{-1/2}$ improvement over the standard method of estimating a detuning parameter.

2. The Susskind-Glogower POVM for phase estimation is defined by

$$F(\phi)d\phi = |\phi\rangle\langle\phi|d\phi \qquad -\pi \leq \phi \leq \pi$$

where

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\theta}|n\rangle \qquad \text{for} -\pi \leq \theta \leq \pi\ ,$$

where $a^\dagger a|n\rangle = n|n\rangle$.

   (a) Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi|\phi\rangle\langle\phi| = I$$

   (b) Find the phase distribution for a coherent state $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n(\alpha^n/\sqrt{n!})|n\rangle$, where $\alpha$is a complex number, and calculate the Fisher information.

3. The interaction picture Hamiltonian for a controllable coupling between two cavity modes with annihilation operators a, b is

$$H_I = \hbar(\mathcal{E}_c(t)ab^\dagger + \mathcal{E}^*{}_c(t)a^\dagger b$$

where $\mathcal{E}_c(t)$ is a classical control field. If mode-a is coherently driven and damped, we can replace a by its steady state average value in the absence of coupling

$$a \to \alpha = \frac{-iE}{\gamma/2 + i\delta}$$

where γ, δ are, respectively, the linewidth and detuning of cavity-a and E is the driving strength in units of frequency.

(a) Show that the Heisenberg equations of motion for mode-b is

$$\frac{db}{dt} = -i\alpha\mathcal{E}(t) \ .$$

(b) Show that the solution for the mean value may be written as

$$\langle b(t)\rangle = -i\alpha\int_{-\infty}^{t} dt' \ \mathcal{E}(t') \ ,$$

on the assumption that in the remote past ($t \to -\infty$) the average value was zero.

(c) Let the time dependence of the control field be given by,

$$\mathcal{E}(t) = \sqrt{\kappa}e^{-\kappa|t|} \ .$$

Calculate the mean photon number,$\langle b^\dagger(t)b(t)\rangle$, in cavity-$b$.

4. A two-level atom, moving in the $z-$direction, with momentum $\hbar k$, is subject to a Raman transition described by the following interaction Hamiltonian.

$$U(\theta) = \exp\left[-i\frac{\theta}{2}(e^{-i\Delta\hat{z}}\sigma_+ + e^{i\Delta\hat{z}}\sigma_-)\right]$$

Show that the joint state of the centre of mass and internal degress of freedom are transformed as

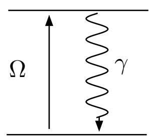
$$\begin{aligned} U(\theta)|k,1\rangle &= \cos\theta/2|k,1\rangle - i\sin\theta/2|k+\Delta,2\rangle \\ U(\theta)|k,2\rangle &= \cos\theta/2|k,2\rangle - i\sin\theta/2|k-\Delta,1\rangle \end{aligned}$$

5. **The following question should be submitted for assesment.**

A Zeeman-split magnetic hyperfine transition is driven by a transverse AC magnetic field. The interaction picture Hamiltonian is

$$H = \hbar\frac{\Delta(B)}{2}\sigma_z + \Omega\sigma_x \ , \qquad (1)$$

where $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ and $\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|$, $\Delta(B)$ is the Zeeman splitting as a function of the external DC magnetic field and Ω is the Rabi frequency of the transverse driving field. The spontaneous decay rate on the transition is γ. The



steady state average of $\langle\sigma_z\rangle$ is given by

$$\langle\sigma_z\rangle_{ss} = -\frac{\gamma^2 + 4\Delta(B)^2}{\gamma^2 + 4\Delta(B)^2 + 8\Omega^2} \ . \tag{2}$$

We assume that $\Delta(B) = \kappa B$.

(a) Calculate the steady state probability to find the atom in the ground state.

(b) The absorption coefficient for AC field is proportional to the steady-state probability to find the system in the ground state, P_g. Describe how this system could be used as a sensor for the static magnetic field at the atom.

(c) Show that to operate as a linear sensor we should choose a fixed external DC bias field given by

$$B_0 = \pm\frac{\sqrt{8\Omega^2 + \gamma^2}}{\sqrt{3}} \ .$$